

# On a Condition for the Nonsingularity of a Matrix

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**Abstract.** A new derivation of a nonsingularity condition is given and related to conditions for the convergence of a matrix iteration.

Gudkov [2] has recently published a condition, in the form of a system of inequalities, for the nonsingularity of a matrix, and Bailey and Crabtree [1] have extended this result by obtaining bounds for the determinant. They point out that the inequalities are consequences of a result due originally to Mehmke and Nekrasov, and further discussed by Ostrowski [5]. The original setting was in terms of the convergence of the Gauss-Seidel iteration. There might be some interest in returning to this original setting and rederiving these inequalities in somewhat more compact form.

The notation will be that used by the author in [3]. Let the matrix under consideration be

$$A = D - B,$$

where  $D$  is null off the diagonal and  $B$  is null on the diagonal. It is required that  $D$  be nonsingular, whence there is no restriction in supposing  $D = I$ . Let  $B$  be separated into its (proper) lower and upper components, whence

$$A = I - L - R.$$

It is known that

$$\delta(I - |B|) \leq \delta(I - B),$$

whence there is no restriction in supposing  $B \geq 0$ , and this will be done. If  $H = (I - L)^{-1}R$ , Stein and Rosenberg ([6], see also [3]) have shown that one and only one of the three possibilities can hold:

$$\rho(H) < \rho(B) < 1, \quad \rho(H) = \rho(B) = 1, \quad \rho(H) > \rho(B) > 1.$$

In the first instance, both the Jacobi and the Gauss-Seidel iterations converge, otherwise neither converges. Hence, for  $A$  to be nonsingular it is necessary and sufficient that  $\rho(H) < 1$ . Evidently,  $H \geq 0$ , whence  $He < e$  is a sufficient condition for this to hold. But these  $n$  inequalities are those of Gudkov.

To see this, it is necessary first to state the Gudkov inequalities for this somewhat simplified form of the problem. They are

$$(1) \quad \begin{aligned} \rho_1 &= \sum \rho_{1j} < 1, \\ \rho_i &= \sum \lambda_{ij} \rho_j + \sum \rho_{ij} < 1, \quad i = 2, 3, \dots, n. \end{aligned}$$

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It will be shown by an induction on the index  $i$  that these inequalities are identical with the inequalities

$$(2) \quad He < e.$$

For  $i = 1$ , it is obvious. Suppose it is verified for  $i \leq p$ , and let  $L_1$  represent the leading principal submatrix of  $L$  of order  $p$ . Let  $R_1$  likewise represent the leading principal submatrix of  $R$  of order  $p$ , and let  $(R_1, R_2)$  represent the first  $p$  rows of  $R$ . Further, let  $e_1$  and  $e_2$  represent (deviating slightly from the notation in [3]), vectors of unit elements. Then, the hypothesis of the induction is that the first  $p$  inequalities in (1) are representable by

$$(3) \quad (I - L_1)^{-1}(R_1 e_1 + R_2 e_2) < e_1.$$

Now, adjoin a row and column to  $I - L_1$ , and a row to  $(R_1, R_2)$ . The first  $p + 1$  conditions in (2) are

$$\begin{bmatrix} I - L_1 & 0 \\ -l' & 1 \end{bmatrix}^{-1} \begin{bmatrix} R_1 & R_2 \\ 0 & r' \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} < \begin{bmatrix} e_1 \\ 1 \end{bmatrix}.$$

Evidently,

$$\begin{bmatrix} I - L_1 & 0 \\ -l' & 1 \end{bmatrix}^{-1} = \begin{bmatrix} (I - L_1)^{-1} & 0 \\ l'(I - L_1)^{-1} & 1 \end{bmatrix},$$

whence the condition that follows those in (3) is

$$(4) \quad l'(I - L_1)^{-1}(R_1 e_1 + R_2 e_2) + r' e_2 < 1.$$

But by the hypothesis of the induction, the first part of (4) coincides with the first summation on the right of (1), and the second part with the second summation. This completes the proof. To obtain from (1) the precise form used by Gudkov, one has only to introduce absolute value signs; multiply by  $D$  and replace  $\delta, \rho_i$  by  $\rho_i$ ; place  $\delta_i$  on the right of the inequalities in (1); and modify the notation.

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